

Lévy walks and generalized stochastic collision models

E. Barkai and V. N. Fleurov

Beverly and Raymond Sackler Faculty of Exact Sciences, School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel

(Received 25 March 1997)

A stochastic collision model is studied in which a test particle of a mass M collides with bath particles of another mass m . If the distribution of time intervals between the collisions is long tailed, the relaxation of momentum of the test particle is algebraic. The diffusion is enhanced and a superdiffusion is characteristic of the test particle motion for long times. It is shown that for long times $\langle x^2(t) \rangle$ is independent of the mass ratio $\epsilon = m/M$. The mass ratio is an important parameter controlling a transition time before which $\langle x^2 \rangle \sim t$ and after which diffusion is enhanced. Special attention is given to the Rayleigh limit where ϵ is small. It is shown that when $\epsilon = 1$ our results are identical to those obtained within the framework of the Lévy walk model. [S1063-651X(97)07511-9]

PACS number(s): 05.40.+j, 02.50.-r, 05.60.+w

I. INTRODUCTION

Anomalous Brownian motion, which is characterized by the mean square displacement of a test particle behaving as

$$\langle x^2(t) \rangle \sim t^\delta, \quad \text{with } \delta \neq 1 \quad (1)$$

is a well known phenomenon [1–4], observed in many physical systems. The exponent $\delta \neq 1$ indicates that the conditions ensuring the validity of the central limit theorem (CLT) are not satisfied. Lévy and Khintchine [5] proposed a generalization of the CLT to the case where x_i in the sum $X_N = \sum_{i=1}^N x_i$ are independent, identically distributed random variables with a long tailed distribution, the existence of the first two moments being not necessarily assumed. Then the random walk X_N is called a Lévy flight. However, this $\langle X_N^2 \rangle$ diverges for all $N > 0$ and, hence, such a Lévy flight cannot represent an anomalous diffusion of the type Eq. (1).

Shlesinger *et al.* [6] have introduced a concept of Lévy walks when a velocity is attributed to each step and the number of steps in the walk is a random variable, and then obtained a finite mean square displacement. A Lévy walk description of chaotic diffusion in Josephson junctions was considered by Geisel *et al.* [7]. They have shown that deterministic maps can produce Lévy walks. Similar modified Lévy flights and walks are often used to model anomalous diffusion [8,9].

A description of a classical Brownian motion is provided by stochastic collision models (see, e.g., [10]) introduced at the turn of the last century by Rayleigh [11] and Drude [12] in order to describe normal ($\delta = 1$) diffusion and transport. Briefly, the Rayleigh model considers a one dimensional heavy test particle with a mass M colliding with light bath particles with a mass m , whereas in the Drude model a particle is scattered with a rate $1/\bar{\tau}$ independent of the mechanical state of the test particle. A common feature of the classical models is that all moments of the probability density function (PDF) of the waiting times between collision events exist.

More recently, the Lorentz gas in which a particle is reflected by equally spaced static spherical obstacles with infinite mass, centered on a hypercubic lattice, was considered, see [13,4], and references therein. It was found that very long trajectories exist along which the particle can move freely. The PDF of the path length l was estimated, basing upon simple geometrical reasoning [14,15], to behave as

$$p(l) \sim \frac{1}{l^3}.$$

Since in this model the particle moves with a constant velocity between collisions the PDF of collision times t is

$$\psi(t) \sim \frac{1}{t^3}.$$

Assuming that the process is renewed after each collision (i.e., neglecting correlations between the successive collisions) the anomalous behavior

$$\langle x^2(t) \rangle \sim t \ln(t)$$

was found. This result is compatible with numerical simulations.

Now, a century after the works of Rayleigh and Drude when the anomalous diffusion draws a special interest, a generalization of classical collision models may provide a new venue to attack this problem. Our aim here is to demonstrate how this type of approach can produce an anomalous diffusion and to follow the relation between it and the Lévy-walk approach. Conditions when the two approaches map one onto another will be emphasized, as well as the situations when they differ.

For normal diffusion an important control parameter is the mass ratio

$$\epsilon \equiv \frac{m}{M}.$$

When $\epsilon \neq 1$ the momentum relaxation to the equilibrium Maxwell distribution is achieved only after the test particle encountered many collisions with the bath particles. This means that a correlation exists between the momentum of the test particle just after the n th kick and the momentum after the $(n+k)$ th kick. This aspect does not exist in the Lévy-walk model. There a particle moves with a velocity which changes at random times, the different velocities being statistically independent.

For the classical models a special interest is the Rayleigh limit when $\epsilon \rightarrow 0$ and $\bar{\tau} \rightarrow 0$ leaving their ratio finite. In such a process the collisions are weak though frequent and the process can be described by a Fokker-Planck equation. This limit is well investigated for the normal diffusion, however, little is known about such a limit in an anomalous case. Using a generalized collision model which assumes the renewal property of the process and a power law behavior of the PDF of the waiting times we investigate among other things the ϵ dependence of the anomalous diffusion. A special attention is paid to the Rayleigh limit.

II. THE MODEL

A particle of a mass M is considered which moves dynamically in one dimension according to the Newton law of motion. At random times the velocity of the particle is randomly changed due to an elastic collision with bath particles of a mass m . The waiting times between the collisions are assumed to be independent identically distributed random variables and thus the number of collisions in a given time is a renewal process [16]. Each elastic impact causes a change of the test particle momentum

$$p^+ = \mu_1 p^- + (1 - \mu_1) \tilde{p}, \quad (2)$$

where

$$\mu_1 \equiv \frac{1 - \epsilon}{1 + \epsilon},$$

p^- (p^+) stands for the momentum value just before (after) the collision, \tilde{p} is the momentum of the incident bath particle. The time of interaction between the test and bath particles is assumed to be negligible relative to all other times in the model. The process is characterized by the PDF of the waiting times between collisions, $q(\tau)$, and by the PDF of the incident bath particle momenta, $f_m(\tilde{p})$.

A common approach is to assume that the renewal process is a Poissonian (as in the Drude model) and that the momenta of the bath particles are Maxwell distributed and characterized by a temperature T (cf., [17]). According to the CLT the diffusion of the test particle is then normal [$\delta=1$ in Eq. (1)]. In the special case when $\epsilon=1$, the test particle momentum is resampled from the Maxwell distribution after each collision. For this case the model becomes the well known strong collision model [18] investigated in the context of condensed matter physics, plasma physics, and chemical reaction rate theory (for references see, e.g., [17]). Another well known limit of this model is that of weak (small ϵ) but frequent collisions [17]. Under certain conditions this limit leads to the well known Fokker-Planck or Kramers equations.

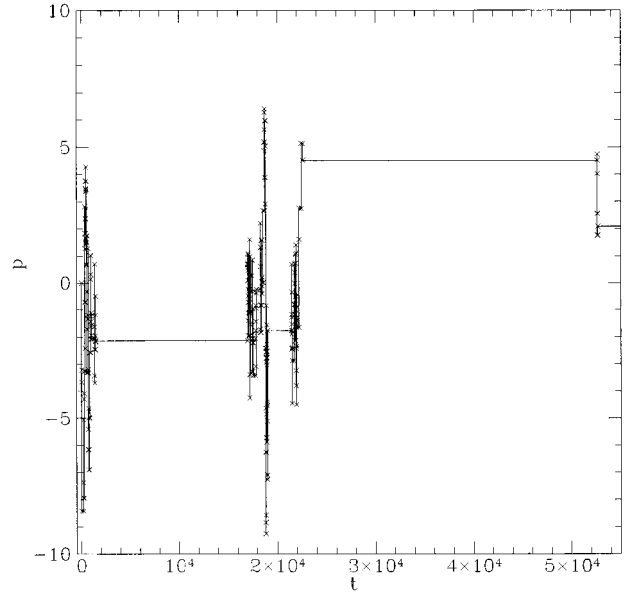


FIG. 1. The momentum of the test particle vs time (dimensionless units). The test particle experienced 200 collisions. The bath particle momentum was drawn from the Maxwell distribution with a variance equal to one. Notice the long time intervals in which no collisions take place.

Sometimes the waiting times between the collisions are assumed to be constant and so $q(\tau) = \delta(\tau - \tau_0)$. When $f_m(\tilde{p})$ is Maxwellian and $\epsilon=1$ the model is used for numerical simulations of complex systems in thermal equilibrium [19]. Zanette and Alemany [20] (see also [21]) chose $\epsilon=1$ and the momentum PDF, $f_m(\tilde{p})$, with a long tail as suggested by Tsallis [22] in the context of nonextensive statistical mechanics. The diffusion is then a Lévy flight and a relation between diffusivity and a generalized temperature is obtained.

The model considered in this paper assumes a Maxwell distribution for the incident particle momenta, remaining thus, within the framework of intensive statistical mechanics. However, the waiting time PDF may be chosen in such a way that its second or even first moment diverges. Then, as we will see below, the test particle momentum relaxation is characterized by a power law dependence on time.

Figure 1 shows the momentum p of the test particle vs time for a typical realization of our stochastic process. The mass ratio $\epsilon=0.1$ is chosen to be rather small and so the momentum relaxes only after many collision events. We chose $q(\tau)$ to be a long tailed PDF behaving asymptotically as

$$q(\tau) \sim \tau^{-4/3}.$$

This means that all moments of $q(\tau)$ diverge. The result of this choice is that long time intervals exist in which no collision event takes place. In these intervals which are of the order of magnitude of the observation time, the momentum of the particle does not change with time and so the particle motion is ballistic. This can be seen in Fig. 2, which presents the location x of the particle vs the momentum. The particle makes long ballistic flights when collisions do not take place.

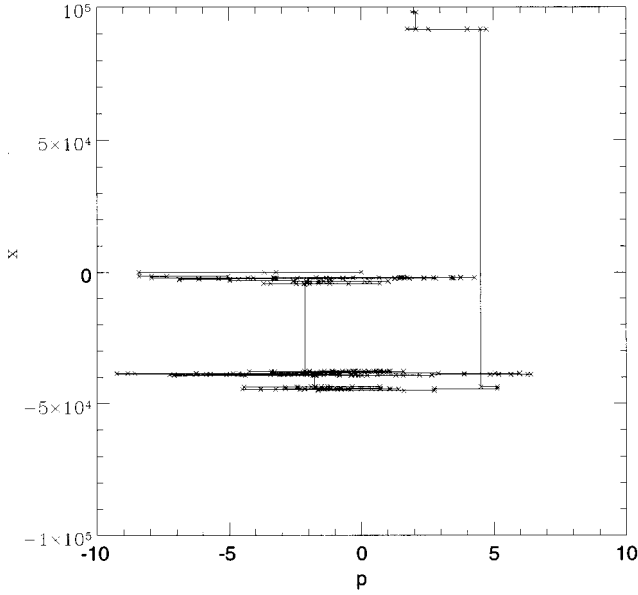


FIG. 2. The phase space for the realization presented in Fig. 1. Long journeys are made during the collisionless time intervals.

III. CHARACTERISTIC FUNCTION

Let $\pi(p, t|p_0, 0)dp$ be the probability that at a time t the test particle momentum lies in the range $(p, p+dp)$ under the initial condition that $p=p_0$ at $t=0$. Then

$$\pi(p, t|p_0, 0)dp = \sum_{s=0}^{\infty} P_s(t) \pi_s(p, t|p_0, 0)dp,$$

where $P_s(t)$ is the probability that the test particle experienced s collisions within the time interval $(0, t)$, and $\pi_s(p, t|p_0, 0)dp$ is the probability that the test particle with the initial momentum p_0 arrives at the momentum range $(p, p+dp)$ after s collisions. Now the characteristic function

$$\langle \exp(ikp) \rangle = \sum_{s=0}^{\infty} P_s(t) \langle \exp(ikp) \rangle_s \quad (3)$$

is defined in which the integral

$$\langle \exp(ikp) \rangle_s = \int dp \exp(ikp) \pi_s(p, t|p_0, 0)$$

can be calculated for the case when the particle moves freely between collision events.

The momentum of the test particle after s collisions is found using Eq. (2),

$$p_s = \mu_1^s p_0 + \sum_{i=1}^s \mu_1^{s-i} (1 - \mu_1) \tilde{p}_i,$$

where \tilde{p}_i is the momentum of the i th incident bath particle. The independent variables \tilde{p}_i , are distributed according to

$$f_m(\tilde{p}) = \frac{1}{\sqrt{2\pi k_b T}} \exp\left(-\frac{\tilde{p}^2}{2mk_b T}\right).$$

Integrating over these variables one arrives at

$$\begin{aligned} \langle \exp(ikp) \rangle_s &= \underbrace{\int \cdots \int}_s \prod_{i=1}^s d\tilde{p}_i f_m(\tilde{p}_i) \exp(ikp_s) \\ &= \exp\left\{ ik\mu_1^s p_0 - \frac{Mk_b T}{2} k^2 (1 - \mu_1^{2s}) \right\}. \end{aligned} \quad (4)$$

The probability $P_s(t)$ is computable once the collision number generating function (CNGF)

$$\psi(z, t) = \sum_{s=0}^{\infty} P_s(t) z^s$$

is known. Inserting Eq. (4) in Eq. (3) and then differentiating the characteristic function with respect to k , the equations

$$\langle p \rangle = p_0 \psi(\mu_1, t)$$

and

$$\langle p^2 \rangle = p_0^2 \psi(\mu_1^2, t) + Mk_b T [1 - \psi(\mu_1^2, t)] \quad (5)$$

relating the moments of the test particle momentum to the CNGF are obtained.

In the Appendix it is shown that if

$$\lim_{t \rightarrow \infty} \psi(\mu_1^r, t) = 0, \quad (6)$$

where r is a positive integer, then

$$\lim_{t \rightarrow \infty} \langle e^{ikp} \rangle = \exp\left(-\frac{Mk_b T}{2} k^2\right), \quad (7)$$

meaning that the particle reaches its thermal equilibrium. We shall show below that the condition Eq. (6) is satisfied also for the PDF, $q(\tau)$, which has no finite moments. Hence, even when no characteristic time scale separating between the microscopic dynamics and the coarse grained macroscopic averaged relaxation exists, the thermal equilibrium is reached.

IV. CNGF

It was shown in Eq. (5) that the CNGF is not only a mathematical tool it also provides a description of the momentum relaxation. In this section we shall investigate this function.

The Laplace transform $\hat{\psi}(z, u)$ of the CNGF is calculated within the framework of the renewal theory [16]. Using the convolution theorem for the Laplace transform it can be shown that

$$\hat{\psi}(z, u) = \frac{1}{u} + \frac{(z-1)\hat{h}(u)}{[1-z\hat{q}(u)]u}. \quad (8)$$

Here $\hat{h}(u)$ is the Laplace transform of the waiting time PDF, $h(\tau)$, for the first collision.

Now we encounter the problem of the initial conditions. In the well known context of continuous time random walks

two types of such conditions are usually considered [2]. The first, stationary type of the initial conditions can be defined if the first moment of the waiting time PDF exists, i.e., the average time $\bar{\tau} \equiv \int_0^\infty q(\tau) \tau d\tau$ between the collisions is finite. Then

$$h_{st}(\tau) = \frac{1 - \int_0^\tau q(t') dt'}{\bar{\tau}} \quad (9)$$

corresponds to a process which can be viewed as going on for a long time before an arbitrarily chosen moment $t=0$, when the observation starts. As for a nonstationary process a customary choice is

$$h_{nst}(\tau) = q(\tau) \quad (10)$$

(even if $\bar{\tau}$ diverges), where the beginning of the observation coincides with the start of the process. Other choices of $h_{nst}(\tau)$ for Eq. (10) could be made as well. However, this choice serves to illustrate the differences between the stationary and nonstationary conditions. Inserting Eq. (9) in Eq. (8), the CNGF for a stationary process is

$$\hat{\psi}_{st}(z, u) = \frac{1}{u} + \frac{(z-1)[1-\hat{q}(u)]}{[1-z\hat{q}(u)]u^2\bar{\tau}} \quad (11)$$

and in the nonstationary case

$$\hat{\psi}_{nst}(z, u) = \frac{1}{u} \frac{1-\hat{q}(u)}{1-z\hat{q}(u)}. \quad (12)$$

It is instructive to investigate the classical case of the Poissonian process, i.e.,

$$\hat{q}(u) = \frac{1}{1+u\bar{\tau}},$$

meaning that

$$\psi_{st}(\mu_1, t) = \psi_{nst}(\mu_1, t) = \exp\left[-(1-\mu_1) \frac{t}{\bar{\tau}}\right].$$

The Rayleigh limit in which $\epsilon \rightarrow 0$ is reached for a given observation time t when

$$\lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} (1-\mu_1)t/\bar{\tau} = \lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} 2\epsilon t/\bar{\tau} = \gamma t, \quad (13)$$

with a finite γ . This means that the average number of collisions $t/\bar{\tau}$ during the time t is very large and even though each collision is very weak a macroscopic relaxation time $1/\gamma$ is well defined since

$$\lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} \psi(t) = \exp(-\gamma t).$$

We see that $\psi(t)$ for finite ϵ and $\psi(t)$ when $\epsilon \rightarrow 0$ have the same functional dependence on time.

Below, the function $\psi(\mu_1, t)$ is investigated for specially chosen functions $q(\tau)$ resulting in anomalous diffusions and slow relaxations. We shall consider the functions

$$\hat{q}(u) \approx \begin{cases} 1 - (Au)^\alpha + c_1(Au)^{2\alpha}, & 0 < \alpha < 1 \\ 1 + (Au)\ln(Au), & \alpha = 1 \\ 1 - \bar{\tau}u + c_\alpha(\bar{\tau}u)^\alpha + c_2(\bar{\tau}u)^2, & 1 < \alpha < 2 \\ 1 - \bar{\tau}u - c_1(\bar{\tau}u)^2 \ln(\bar{\tau}u) + c_2(\bar{\tau}u)^2, & \alpha = 2 \end{cases}$$

and

$$\hat{q}(u) \approx 1 - \bar{\tau}u + c_1(\bar{\tau}u)^2 + c_\alpha(\bar{\tau}u)^\alpha, \quad 2 < \alpha < 3 \quad (14)$$

where c_α , c_1 , and c_2 are dimensionless constants. These PDFs have the property that for $\alpha \leq 1$ all moments diverge, for $1 < \alpha \leq 2$ only the first moment $\bar{\tau}$ converges, while for $2 < \alpha < 3$ the first two moments converge.

First, the two types of initial conditions are considered and it is shown that they lead to different types of relaxation even in the long time limit. In the stationary case

$$\psi_{st}(\mu_1, t) = \frac{c_\alpha}{\Gamma(2-\alpha)} \left(\frac{t}{\bar{\tau}}\right)^{1-\alpha} + \frac{2(\alpha-1)c_\alpha}{\Gamma(2-\alpha)} \left(\frac{1-\epsilon}{2\epsilon}\right) \left(\frac{t}{\bar{\tau}}\right)^{-\alpha} + o(t^\delta), \quad (15)$$

for $1 < \alpha < 3$ with $1/\Gamma(0) = 0$ and $\delta = \max(-\alpha-1, 1-2\alpha)$. The leading term is independent of ϵ and, hence, independent of the collision strength. For large t it is easy to show that $\psi_{st}(\mu_1, t) \approx \xi_{st}(t)$, where

$$\xi_{st}(t) = 1 - \int_0^t h_{st}(\tau) d\tau$$

is the probability that the first collision occurs at a time larger than t . Thus the slow relaxation given in Eq. (15) is due to samples where no collisions occur and that is why the relaxation does not depend on the strength of the collisions.

The Rayleigh limit Eq. (13) is now considered. Using the asymptotic expansion (15) we find

$$\lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} \psi(\mu_1, t) = 0.$$

Clearly this Rayleigh limit fails in the sense that it does not give the correct asymptotic power law behavior found in Eq. (15). Thus unlike the Poissonian case this limit cannot be used to approximate a process for which the mass ratio is small though finite. One may consider a generalization of the Rayleigh limit where

$$\lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} 2\epsilon \left(\frac{t}{\bar{\tau}}\right)^\alpha \equiv \gamma_\alpha t^\alpha$$

is finite. For this case the leading term in Eq. (15) vanishes and we find

$$\lim_{\epsilon \rightarrow 0, t/\bar{\tau} \rightarrow \infty} \psi_{st}(\mu_1, t) = \frac{2(\alpha-1)c_\alpha}{\Gamma(2-\alpha)} \frac{1}{\gamma_\alpha t^\alpha}.$$

However, for any small though finite ϵ this equation does not provide a really full description of the momentum relaxation since according to Eq. (15) there exists a time after which

$$\psi_{st} \sim t^{1-\alpha}.$$

The condition that this behavior may be observed [i.e., the condition that the first term in Eq. (15) is indeed the leading term] reads for small ϵ

$$\frac{t}{\bar{\tau}} \gg \frac{\alpha-1}{\epsilon}.$$

One sees that for $\alpha > 1$ taking the limit $\epsilon \rightarrow 0$ alters the functional dependence of momentum relaxation. Therefore a process in which ϵ is finite cannot be approximated by another limiting process where $\epsilon \rightarrow 0$.

In the nonstationary case we use Eq. (12) to find

$$\psi_{nst}(\mu_1, t) = \frac{1+\epsilon}{2\epsilon} \frac{1-\alpha}{\Gamma(2-\alpha)} \left(\frac{t}{A}\right)^{-\alpha} + o(t^{-2\alpha})$$

for $\alpha \leq 1$ and

$$\psi_{nst}(\mu_1, t) = \frac{c_\alpha(\alpha-1)}{\Gamma(2-\alpha)} \frac{1+\epsilon}{2\epsilon} \left(\frac{t}{\bar{\tau}}\right)^{-\alpha} + o(t^{-\alpha-1})$$

for $1 < \alpha < 3$. The amplitude of the leading term depends on ϵ and the power law differs from that in the stationary case. The leading term for the nonstationary case has the same power law dependence as the second term in Eq. (15). The difference between the two ensembles has to do with the fact that for the stationary case, the first waiting time $\langle \tau h_{st}(\tau) \rangle$ is in the statistical sense much longer than the first waiting time in the nonstationary case. In fact, it is infinite for $1 < \alpha \leq 2$, so the test particle waits on the average an infinite time until it experiences the first collision.

For the case $\alpha \leq 1$, $\bar{\tau}$ does not exist and clearly one cannot consider the Rayleigh limit (13). However, one may consider a generalization

$$\lim_{\epsilon \rightarrow 0, (t/A)^\alpha \rightarrow \infty} 2\epsilon \left(\frac{t}{A}\right)^\alpha = \beta_\alpha t^\alpha. \quad (16)$$

Then

$$\lim_{\epsilon \rightarrow 0, (t/A)^\alpha \rightarrow \infty} \psi_{nst}(\mu_1, t) = \frac{(1-\alpha)}{\Gamma(2-\alpha)} \frac{1}{\beta_\alpha t^\alpha},$$

showing that this limiting procedure does not alter the time dependence of the CNGF found for the finite ϵ case. Thus we see that if all moments of a PDF, $q(\tau)$, exist the Rayleigh limit (13) works well. When the first moment of the PDF exists but higher moments diverge, the limit $\epsilon \rightarrow 0$ fails, while when all moments of the PDF diverge the generalized Rayleigh limit (16) can be used.

V. MEAN SQUARE DISPLACEMENT

Now the mean square displacement $\langle \hat{x}^2(t) \rangle$ is calculated by means of the relation

$$\langle \hat{x}^2(u) \rangle = \frac{2\hat{C}(u)}{M^2 u^2}$$

for its Laplace transform, in which $\hat{C}(u)$ is the Laplace transform of the correlation function

$$C(t) \equiv \langle p(0)p(t) \rangle = k_b T M \psi_{st}(\mu_1, t), \quad (17)$$

calculated under the stationary initial conditions. Then one obtains for large times

$$\begin{aligned} \langle x^2(t) \rangle &= \frac{2k_b T}{M} \left\{ \frac{c_\alpha \bar{\tau}^2}{\Gamma(4-\alpha)} \left(\frac{t}{\bar{\tau}}\right)^{3-\alpha} + \left(c_2 + \frac{1-\epsilon}{2\epsilon}\right) \bar{\tau} t \right\} \\ &\quad + o(t^{2-\alpha}) \quad \text{for } 1 < \alpha < 2, \\ \langle x^2(t) \rangle &= \frac{2k_b T}{M} \left\{ c_1 \bar{\tau} \ln\left(\frac{t}{\bar{\tau}}\right) + \left(c_2 + \frac{1-\epsilon}{2\epsilon}\right) \bar{\tau} t \right\} \\ &\quad + o[\ln(t)] \quad \text{for } \alpha = 2, \end{aligned} \quad (18)$$

$$\langle x^2(t) \rangle = \frac{2k_b T}{M} \left(c_1 + \frac{1-\epsilon}{2\epsilon} \right) \bar{\tau} t + o(t^{3-\alpha}) \quad \text{for } 2 < \alpha < 3.$$

Time exponents of the leading terms are the same as those found within the framework of Lévy-walk models [24]. For $\alpha \leq 2$ the diffusion is enhanced and according to our result Eq. (15) and discussion it is not surprising that the first leading term in Eq. (18) is independent of ϵ . The linear correction term becomes increasingly important as ϵ becomes smaller. The superdiffusive term exceeds the linear one only when the time t becomes large enough,

$$\frac{t}{\bar{\tau}} \gg \left[\frac{1}{2\epsilon} \frac{\Gamma(4-\alpha)}{c_\alpha} \right]^{1/(2-\alpha)}, \quad 1 < \alpha < 2. \quad (19)$$

This time can become especially long if α approaches 2 from below. Thus, although ϵ does not control the enhanced diffusion it is an important parameter controlling the transition from the normal to superdiffusive behavior.

For a Poisson waiting time distribution the process can be described by a master equation and a Fokker-Planck equation holds in the Rayleigh limit (13). The mean square displacements in both descriptions increase linearly with time for long times. This limit in Eq. (18) results also in a linear time dependence $\langle x^2(t) \rangle = 2Dt$ with

$$D = \frac{k_b T}{M \gamma}$$

and γ defined in Eq. (13). Actually the convergence is not uniform, meaning that for any small but finite ϵ there is a large enough time t after which the superdiffusive term becomes larger than the linear one. Therefore both terms are necessary for a proper description of $\langle x^2(t) \rangle$ for $1 < \alpha \leq 2$ close to the Rayleigh limit.

For completeness, we specify $\langle x^2(t) \rangle_{\text{nst}}$ for the nonstationary case using in calculations the method given in [23]. Then for different values of the parameter α one obtains the equations

$$\langle x^2(t) \rangle_{\text{nst}} \simeq \frac{k_b T}{M} (1 - \alpha) t^2 \quad \text{for } 0 < \alpha < 1, \quad (20)$$

$$\langle x^2(t) \rangle_{\text{nst}} \simeq \frac{k_b T}{M} \frac{t^2}{\ln\left(\frac{t}{A}\right)} \quad \text{for } \alpha = 1,$$

$$\langle x^2(t) \rangle_{\text{nst}} \simeq \frac{2k_b T}{M} \frac{c_\alpha \bar{\tau}^2 (\alpha - 1)}{\Gamma(4 - \alpha)} \left(\frac{t}{\bar{\tau}}\right)^{3 - \alpha} \quad \text{for } 1 < \alpha < 2$$

for the leading terms, all these being independent of ϵ . For $0 < \alpha < 1$ the motion is ballistic. When $2 \leq \alpha < 3$, $\langle x^2(t) \rangle_{\text{nst}}$ coincides with the leading term in Eq. (18).

VI. COMPARISON WITH THE LÉVY-WALK MODEL

Now our results are compared with the results obtained within the framework of the Lévy walk [7,24,25] which considers a velocity model, in which the test particle moves at a constant velocity for a given time, until it changes direction (without changing the magnitude). The velocity model describes well the dynamical properties generated by iterating deterministic chaotic maps. The velocity model is similar to a strong collision limit. Indeed, assuming $k_b T/M = 1$ and $c = 1$, Eqs. (11) and (17) of our model produce

$$\langle \hat{x}^2(u) \rangle = 2 \left[\frac{1}{u^3} - \frac{1 - \hat{q}(u)}{u^4 \bar{\tau}} \right],$$

which coincides with Eq. (29) in [25]. The two models have the same asymptotic behaviors of $\langle x^2(t) \rangle$ which in our model is independent of ϵ for large times. The sources of fluctuations are long time intervals when no collisions occur and these in turn are independent of the strength of collisions.

VII. SUMMARY

This work generalizes the classical models of Rayleigh and Drude and incorporates waiting time PDF with divergent moments. It results in anomalous superdiffusive behavior of the test particle. We have found that for long times the diffusion is independent of the mass ratio ϵ . This differs strongly from the classical models where the diffusion constant D exists and is ϵ dependent. The corrections to the long time Lévy-walk-type behavior of $\langle x^2(t) \rangle$ were found. We show that when ϵ becomes smaller these corrections become of greater importance for longer times.

A special attention was given to the Rayleigh limit. It was shown that in this limit the time dependence of the relaxation and the diffusion may differ from the exact behavior valid for finite ϵ . A generalization (16) of the Rayleigh limit seems to work well for the case when the PDF $q(\tau)$ has no finite moments.

ACKNOWLEDGMENTS

We are grateful to J. Klafter, E. Pollak, and Z. Schuss for fruitful discussions.

APPENDIX

To prove Eq. (7) we shall consider first the n th moment of the mechanical momentum $\langle p^n \rangle$. We find it convenient to define, using Eq. (4),

$$\langle e^{ikp_s} \rangle = \exp\left(-\frac{Mk_b T}{2} k^2\right) e_{\mu_1}(k), \quad (A1)$$

with

$$e_{\mu_1}(k) \equiv \exp\left(i\mu_1^s k p_0 + \frac{Mk_b T}{2} \mu_1^{2s} k^2\right).$$

Then inserting Eq. (A1) in Eq. (3) and differentiating n times we find

$$\begin{aligned} \langle p^n \rangle &= \frac{d^n}{dk^n} \langle e^{ikp} \rangle \Big|_{k=0} \\ &= \left\{ \left[\frac{d^n}{dk^n} \exp\left(-\frac{Mk_b T}{2} k^2\right) \right] \sum_{s=0}^{\infty} P_s(t) e_{\mu_1}(k) + \dots + \binom{n}{r} \right. \\ &\quad \times \left. \left[\frac{d^{n-r}}{dk^{n-r}} \exp\left(-\frac{Mk_b T}{2} k^2\right) \right] \left[\sum_{s=0}^{\infty} P_s(t) \frac{d^r}{dk^r} e_{\mu_1}(k) \right] + \dots \exp\left(-\frac{Mk_b T}{2} k^2\right) \left[\sum_{s=0}^{\infty} P_s(t) \frac{d^n}{dk^n} e_{\mu_1}(k) \right] \right\} \Big|_{k=0}. \quad (A2) \end{aligned}$$

The first term on the right hand side of this equation is considered. Since by definition $e_{\mu_1}(0) = 1$ and using the normalization condition $\sum_{s=0}^{\infty} P_s(t) = 1$, we find

$$\left[\frac{d^n}{dk^n} \exp\left(-\frac{Mk_b T}{2} k^2\right) \right] \Big|_{k=0} \sum_{s=0}^{\infty} P_s(t) e_{\mu_1}(0) = \frac{d^n}{dk^n} \exp\left(-\frac{Mk_b T}{2} k^2\right) \Big|_{k=0}. \quad (A3)$$

It is clear that this term gives the thermal equilibrium result. We now consider the other terms in Eq. (A2) satisfying the condition $1 \leq r \leq n$. Using the change of variables $\mu_1^s k = \alpha$ we find

$$\sum_{s=0}^{\infty} P_s(t) \frac{d^r}{dk^r} e_{\mu_1}(k) \Big|_{k=0} = \left[\frac{d^r}{d\alpha^r} \exp\left(i\alpha p_0 - \frac{Mk_b T}{2} \alpha^2 \right) \right] \Big|_{\alpha=0} \psi(\mu_1^r, t).$$

Inserting this equation in the finite sum Eq. (A2) we see that condition Eq. (6) nullifies all terms in Eq. (A2) except for the first term, Eq. (A3). We find

$$\lim_{t \rightarrow \infty} \langle p^n \rangle = \frac{d^n}{dk^n} \exp\left(-\frac{Mk_b T}{2} \right) \Big|_{k=0},$$

meaning that the characteristic function is indeed the Gaussian Eq. (7).

-
- [1] E. W. Montroll and M. F. Shlesinger, in *Nonequilibrium Phenomena II, From Stochastics To Hydrodynamics*, edited by J. L. Lebowitz and E. W. Montroll (North-Holland, Amsterdam, 1984).
- [2] J. W. Haus and K. W. Kehr, *Phys. Rep.* **150**, 263 (1987).
- [3] S. Havlin and D. Ben Avraham, *Adv. Phys.* **36**, 695 (1987).
- [4] J. P. Bouchaud and A. Georges, *Phys. Rep.* **195**, 127 (1990).
- [5] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables* (Addison-Wesley, Reading, MA, 1968).
- [6] M. Shlesinger, B. West, and J. Klafter, *Phys. Rev. Lett.* **58**, 1100 (1987).
- [7] T. Geisel, J. Nierwetberg, and A. Zacherl, *Phys. Rev. Lett.* **54**, 616 (1985).
- [8] *Lévy Flights and Related Topics in Physics*, edited by M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch (Springer-Verlag, Berlin, 1994).
- [9] J. Klafter, M. F. Shlesinger, and G. Zumofen, *Phys. Today* **49**, 33 (1996).
- [10] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
- [11] M. R. Hoare, *Adv. Chem. Phys.* **XX**, 135 (1971).
- [12] N. W. Ashcroft and N. D. Mermin, *Solid State Physics* (Saunders College Publishing, Forth Worth, 1976).
- [13] H. van Beijeren, *Rev. Mod. Phys.* **54**, 195 (1982).
- [14] J. P. Bouchaud and P. Le Doussal, *J. Stat. Phys.* **41**, 225 (1985).
- [15] A. Zacherl, T. Geisel, J. Nierwetberg, and G. Radons, *Phys. Lett.* **114A**, 317 (1986).
- [16] D. R. Cox, *Renewal Theory* (Methuen & Co., London, 1970).
- [17] E. Barkai and V. Fleurov, *Phys. Rev. E* **52**, 1558 (1995).
- [18] R. Kubo, M. Toda, and N. Hashitsume, *Statistical Physics 2* (Springer-Verlag, Berlin, 1991).
- [19] E. Barkai and V. Fleurov, *Phys. Rev. E* **52**, 137 (1995).
- [20] D. H. Zanette and P. A. Alemany, *Phys. Rev. Lett.* **75**, 366 (1995).
- [21] C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, *Phys. Rev. Lett.* **75**, 3589 (1995).
- [22] C. Tsallis, *J. Stat. Phys.* **52**, 479 (1988).
- [23] E. Barkai and V. Fleurov, *Chem. Phys.* **212**, 69 (1996).
- [24] G. Zumofen and J. Klafter, *Phys. Rev. E* **47**, 851 (1993).
- [25] G. Zumofen and J. Klafter, *Physica D* **69**, 436 (1993).